Chattering Free Sliding Modes in Robotic Manipulators
Control

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Abstract: In this paper the sliding mode motion design is considered for the nonlinear plants which are linear with respect to the control input. The dynamics of the robotic manipulators is treated with and without the dynamics of the actuators. When the dynamics of the actuators is included a design of the sliding modes for the systems with discontinuous control is performed. If actuators’ dynamics is neglected the control is assumed to be continuous quantity. By combining the variable structure systems and Lyapunov designs a new algorithm is developed which posses all good properties of the sliding mode systems while avoiding unnecessary discontinuity of the control and thus eliminating chattering. Neither the explicit calculation of the equivalent control nor high gain inside the boundary layer are used. The parameters of the control depend on the plant’s gain matrix, and the gradients of the sliding mode manifold. This control method is then applied to develop a unified control strategy for the motion control systems including the path tracking control, the impedance control and the force control of a robotic manipulator. It is shown that all these tasks can be formulated in the same mathematical form in which selected so-called sliding mode functions, must track their references. In this way the systems state is forced to remain on the selected manifold in the state space after reaching it. The solution is interpreted in both the Joint space and the Work space for n-degrees of freedom robotic manipulator.

1 Introduction

In the control of robotic manipulators trajectories are often modified by contact forces or tactile stimuli occurring during the motion. This means that the integration of the position control and force control systems of a robotic manipulator is of paramount importance in many applications. Compliance control describes the wide variety of the control structures [1,2,3] aimed to adapt trajectories during
the motion on the basis of the information about forces occurring as a result of the contact with environment.

The application of the sliding modes in the electrical machines and robotic manipulator control is widely discussed in many papers [4,5,6,7]. It was reported in [4] that sliding modes in motion control exhibit unwanted motion (so called chattering) imposed by the discontinuity of the control action. In robotics application, the control plant, a robotic manipulator with torque vector as the control input, has continuous control input and discontinuity of the control is artificially introduced. As the result the acceleration becomes discontinuous and consequently velocity exhibits high frequency oscillations - so called chattering. On this ground it was concluded that chattering is the phenomena which cannot be avoided in the framework of the systems with discontinuous control and that some changes in the design methods should be introduced.

Many different ideas were proposed to deal with that problem. Two basic approaches can be distinguished. So called "boundary layer" method [4] is nothing more than the attempt to apply the high gain feedback when the system’s motion reaches ϵ-vicinity of a sliding mode manifold. This approach is based on the idea of the equivalence of the high gain systems and the systems with sliding modes [8]. The application of this method is very complicated and a few experimental results with its application were presented. Another idea is based on the calculation of the equivalent control and its application as the control input when system’s state reaches ϵ-vicinity of the sliding mode manifold [5,6,9,10]. The application of this method faces a problem of the abundant information about controlled plant needed to calculate equivalent control. Both ideas are applicable only in the systems in which control actions are continuous in nature, like in motion control systems when torque (or force or current) is treated as the control input.

In many dynamical systems discontinuity of the control is "the way of life", like in electrical machines control, in switching power converters in robotic manipulators when mathematical description is augmented to include the actuators dynamics. In these systems above results are not applicable directly and their application should be carefully studied. The same is true for all systems with Pulse Width Modulated outputs. For these systems classical Sliding Modes (SM) design methods are developed.

In this paper the sliding mode motion design is considered for the nonlinear plants which are linear with respect to the control input. Control is assumed to be continuous quantity. By combining the variable structure systems and Lyapunov designs a new algorithm is developed. It posses all good properties of the sliding mode systems while avoiding unnecessary discontinuity of the control and thus eliminating chattering. Neither the explicit calculation of the equivalent control nor high gain inside the boundary layer are used. The parameters of the control depend on the plant’s gain matrix, and the gradients of the sliding mode manifold. This control method is then applied to develop a unified control strategy for the motion control systems including the path tracking control, the impedance control and the force control of a robotic manipulator. It is shown that all these tasks can be formulated in the same mathematical form in which selected linear or nonlinear combinations of the systems states, so-called sliding mode functions, must track their references. In this way the systems state is forced to remain on the selected manifold in the state space after reaching it. These manifolds are defined by the selection of the sliding mode functions. The desired dynamics is achieved by the selection of the manifolds. Such a formalization of the control problem enables to select the controller of the same structure for common control problems in the motion control systems. The solution is interpreted in both the Joint space and the Work space for n-degrees of freedom robotic manipulator.

Simulation and experimental results are presented to confirm characteristics of the proposed system.
2 The control system design

The design of the control system will be, first, demonstrated for a nonlinear systems presentable in the regular form [11]

\[
\frac{dx_1}{dt} = f_1(x_1, x_2) \tag{1}
\]

\[
\frac{dx_2}{dt} = f_2(x_1, x_2) + B_2(x)u + B_2(x)d(t); \tag{2}
\]

\[x_1 \in \mathbb{R}^{n-m}, \quad x_2 \in \mathbb{R}^m, \quad u \in \mathbb{R}^m, \quad f \in \mathbb{F}^n, \quad \text{rank}(B_2(x)) = m.\]

The components of control input and of the vector \(dx_2/dt\) are assumed bounded.

\[u_i \in [u_{i\text{min}}, u_{i\text{max}}]; \quad (dx_2/dt) \in [\alpha_{i\text{min}}, \alpha_{i\text{max}}], \quad (i = 1, ..., m) \tag{3}\]

The goal is to find control \(u\) such that the motion of the system (1)-(3) is restricted to belong to the manifold \(S\),

\[S = \{x : \varphi(t) - \sigma_a(x) = \sigma(x, t) = 0\}, \tag{4}\]

where \(\sigma_a(x)\) and \(\varphi_a(t)(i = 1, ..., m)\) are continuous functions. Functions \(\varphi_a(t)\) are assumed bounded with bounded first time-derivatives. These functions can be interpreted as the references to be tracked by selected combinations \(\sigma_a(x)\) of the system’s states. In the theory of the VSS with sliding modes it is proven that the closed-loop behavior of the system (1),(2),(3) on manifold (4) is determined by the selection of the manifold [12].

In the following sections the sliding mode design assuming that control is discontinuous will be outlined following results in [12,13]. Then a new design of a continuous control which will ensure the motion of the system (1)-(3) on the manifold (4) like the discontinuous control does will be discussed in details. Comparison with discontinuous case will be established by simulation of the same problem using both methods.

2.1 The discontinuous control design

It has been proven in the theory of variable structure systems with sliding modes, [12] that a discontinuous control

\[u(x) = F(x, t)\text{sig}[\varphi(t) - \sigma_a(x)] = F(x, t)\text{sig}[^{\sigma}(x, t)] \tag{5}\]

provides the existence of the sliding mode motion for the system (1)-(3) on the manifold (4). According to the so-called equivalent control method, a fictitious continuous control is calculated as a solution of the system of the algebraic equations \(d[\varphi(t) - \sigma_a(x, u = u_{eq})]/dt = 0\) and denoted as the equivalent control. Than it should be substituted into system’s description (1), (2) and the equations of motion can be determined as \(dx_1/dt = f_1(x_1, x_2), dx_2/dt = f_2(x_1, x_2) + B_2(x)u_{eq} + B_2(x)d(t)\). For system (1), (2) with control (5) on the manifold (4) with \(d\sigma_a/dt = G_1x_1 + G_2x_2\) where \([\partial\sigma_a/\partial x_1] = G_1, [\partial\sigma_a/\partial x_2] = G_2, \text{rank}G_2 = m\), application of the equivalent control methods gives

\[u_{eq} = -d + (G_2B_2)^{-1}(d\varphi/dt - G_2f_2 - G_1f_1) \tag{6}\]

and consequently the equations of motion can be expressed as

\[
\frac{dx_1}{dt} = f_1(x_1, x_2) \tag{7}
\]
\[ \sigma_a(x) = \varphi(t) \quad (8) \]

The sliding mode exists if \( u_{eq} \) is unique and if \( u_{imin} < u_{ieq} < u_{imax} \), \((i = 1, \ldots, m)\). If such a control can be applied the system’s motion, after initial transient to reach manifold \( S \), will be restricted to belong to the manifold \( S \) and the system’s states will satisfy equation (8). The selection of the functions \( \varphi(t) \) and \( \sigma_a(x) \) plays important role in the closed-loop dynamics. For example \( x_2 \) determined from (8) can be treated as a control input in (7). Equations of motion (7)-(8) describe a \((n - m)\) order dynamical system. They do not depend on control and are determined merely by the properties of the control plant and the equations of the surfaces \( \sigma_i = 0, (i = 1, \ldots, m) \) [12,13].

2.2 The continuous control design

The robustness against parameters variation (complete rejection of all variations in equation (2)) and disturbance, and the fact that the equations of motion (7), (8) on the manifold (4) are of lower order, are attractive properties of the systems with sliding modes but for some plants discontinuity of control is not acceptable. This is true for all motion control systems when forces and/or torques are considered as the control inputs. The control discontinuity in such systems is imposed and in the experimental work it cannot be tolerated nor achieved due to the neglected actuator’s dynamics. That rises the question of the design of the control system which will preserve most of the good features achieved in sliding motion systems, while the discontinuity of the control action is avoided. Such a design is presented in this section.

Similarly to the definition adopted in [10], any motion of the system (1)-(3) that occurs in the \( \epsilon \) -vicinity of the manifold (4), on which the sliding mode motion can exists with control (5), will be referred to as the sliding mode motion. This reflects the essence of the sliding mode design method: the control is selected to ensure the stability of the solution \( \sigma(x, t) = 0 \) i.e. the motion on the manifold (4). Another essential features of the systems with sliding modes is that the reaching time, from any initial conditions to the \( \epsilon \) -vicinity of the manifold \( S \), is finite. In this framework the design of the control system should ensure the stability on the sliding mode manifold, while the selection of the manifold will provide the means to determine the equations of motion. For system (1)-(3) and selected manifold, the following design procedure can be adopted:

- select a Lyapunov function candidate \( v(\sigma) \), such that, if the Lyapunov stability criteria are satisfied, the solution \( \varphi(t) - \sigma_a(x) = 0 \) is stable on the trajectories of the system (1)-(3);
- select the form which the time derivative of the Lyapunov function should satisfy and find control \( u \) such that selected form is achieved on the trajectories of the system (1)-(3) and (4);
- find the equations of motion on the selected manifold with designed control.

This procedure, if successfully completed, will ensure the stability of the solution \( \sigma(x, t) = 0 \) thus the motion of a closed-loop system will remain on the manifold \( S \) after reaching it. Moreover the attractiveness of the manifold is also ensured by the virtue of the selected stability criteria. After reaching the sliding mode manifold the equations of motion must satisfy the equations (7)-(8). The equations of motion in reaching stage must be determined separately. The system’s state will reach any \( \epsilon \)-vicinity \((\epsilon \neq 0)\) of the sliding mode manifold in the finite time interval. In this procedure the solution is merely determined by the selection of the Lyapunov function. This selection should ensure the stability of the projection of the system’s motion in the origin of the subspace whose coordinates are distances from the sliding mode manifold.
The selection of a Lyapunov function is always governed by the requirement that it should be as simple as possible. For system (1)-(3) and selected manifold (4) the first choice is the Lyapunov function as a quadratic form

\[ v = \frac{\sigma^T \sigma}{2} \]  

(9)

The solution \( \sigma(x, t) = 0 \) will be stable if the time derivative of the Lyapunov function can be expressed as

\[ \frac{dv}{dt} = -\sigma^T D \sigma \]  

(10)

where \( D \) is positive definite matrix.

Further calculations will be carried out for sliding mode manifold (4) defined by \( \sigma_a(x) = x_2 + Gx_1 \). Note that any function \( \sigma_a(x) = G_2x_2 + G_1x_1 \) with \( \text{rank}G_2 = m \) can be reduced to \( \sigma_a(x) = G_2(x_2 + Gx_1) \). From \( \frac{dv}{dt} = -\sigma^T d\sigma/dt \) and \( d\sigma/dt = d\varphi/dt - f_2 - B_2u - B_2 d - Gf_1 \) control can be calculated as

\[ u = \text{sat} \left[ -d + (B_2)^{-1}(d\varphi/dt - f_2 - Gf_1) + (B_2)^{-1}D\sigma \right] = \text{sat} \left[ u_{eq} + (B_2)^{-1}D\sigma \right] \]  

(11)

where \( \text{sat}(\bullet) \) is a saturation function with \( u_{min} = \min[\text{sat}(\bullet)] \) and \( u_{max} = \max[\text{sat}(\bullet)] \).

Control (11) differs from the equivalent control by the term \( (B_2)^{-1}D\sigma \) which assures the attractiveness of the sliding mode manifold. This term becomes zero after reaching manifold \( S \). Control (11) is continuous (\( u_{eq} \) is continuous and sliding mode function is continuous by assumption) and chattering is indeed eliminated. To calculate (11) information about the equivalent control is needed, so this solution is not practical. By using equality \( B_2u_{eq} = B_2u + d\sigma/dt \) the result of short algebra can be written as

\[ u(t) = \text{sat} \left[ u(t^-) + (B_2)^{-1}(D\sigma + \frac{da}{dt}) \right], \quad t = t^- + \Delta, \quad \Delta \to 0 \]  

(12)

The value of the control at the instant \( t \) is calculated from the value at the time \( (t - \Delta) \) and the weighted sum of the control error \( \sigma \) and its time derivative \( d\sigma/dt \). Control (12) is continuous function everywhere except in the points of discontinuity of the function \( \sigma(x, t) \). The stability conditions for the selected control must be examined first.

In the discrete time systems with no computational delay the relations between measured and computed variables are as follows: the measurement are taken before the calculation of new control value (denoted as \( \bullet(t^-) \)). The control is calculated without computational delay and all variables taken at the moment renewed control is applied are denoted by \( \bullet(t^+) \). Note that all continuous functions and variables satisfy \( \bullet(t^-) = \bullet(t^+) \). By taking into account these relations, (12) can be rewritten in the following form \( u(t^+) - u(t^-) = (GB)^{-1}(D\sigma(t^-) + d\sigma(t^-)/dt) \). By calculating the derivative of function \( \sigma \) at \( t = t^+ \) one can find the following \( d\sigma(t^+)/dt = d\varphi(t^+)/dt - Gf_1(t^+) - (f_2(t^+) + B_2 d(t^+)) - B_2 u(t^+) \). By substituting the control input at \( t = t^+ \), one can find \( d\sigma(t^+)/dt = d\varphi(t^-)/dt - Gf_1(t^-) - (f_2(t^-) + B_2 d(t^-)) - B_2 u(t^-) + B_2^{-1}(D\sigma(t^-) + d\sigma(t^-)/dt) \) which can be easily transformed to \( d\sigma(t^+)/dt = d\sigma(t^-)/dt - (D\sigma(t^-) + d\sigma(t^-)/dt) = -D\sigma(t^-) = D\sigma(t^+) \). Time derivative of the Lyapunov function, at \( t = t^+ \), can be expressed as

\[ \frac{dv(t^+)}{dt} = -\sigma(t^+)^T D \sigma(t^+) \]  

(13)

This shows that for the systems without computational delay the stability criterion is satisfied, everywhere except of the discontinuity points of \( \sigma(x, t) \).

In these points \( u \) takes either maximum or minimum value so by the assumption that the time derivatives of the state coordinates are limited the stability criteria is satisfied. If \( D \) is positive definite
matrix the time derivative of the Lyapunov function is negative definite, i.e. solution $\sigma(x, t) = 0$ is stable and the motion of the system from arbitrary initial conditions will reach manifold $S$. Control (12) is continuous and chattering is indeed eliminated.

In this design the control algorithm essentially depends on the form of the Lyapunov function and its derivative. Previous selection of the Lyapunov function is governed by the task to find simple expression for the control, possibly independent of the full knowledge of the system’s dynamics. For example by selecting the same Lyapunov function as in (9) and requiring that its time derivative has the form $\frac{dv}{dt} = -\sigma^T D \text{sig}(\sigma), D$ positive definite matrix, the application of the above procedure gives the control $u(t) = u(t - \Delta) + B^1 D \text{sig}(\sigma)$. It is easy to prove that this control satisfies required relations and ensure finite reaching time.

The motion of the system (1), (2) with control (12) has three stages. First stage is along the trajectories with $u = u_{\text{min}}$ or $u = u_{\text{max}}$. This stage is the same as for the system with discontinuous control - the trajectories are determined by the parameters of the system. Second stage is along the trajectories $u_{\text{min}} < \text{sat}(\cdot) < u_{\text{max}}$ and is governed by the model

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$

$$\frac{d\sigma}{dt} + D\sigma = 0$$

These equations describe $n-$th order dynamical system. By comparison with the sliding mode equations (7)-(8) it can be observed that: equation (14) is the same as (7), equation (15) consists of $m$ first-order differential equations that describe the change of the distances from the manifold $S$. These distances decay with a rate determined by the elements of matrix $D$. For the systems represented in the controller canonical form the equations of motion will be reduced to the set (15). These equations do not depend on the control and the motion is merely determined by matrices $D$ and $G$. If matrix $D$ is selected diagonal then (15) splits in $m$ independent first order differential equations.

The third stage of the motion is along the trajectories that belong to the in the $\varepsilon$-vicinity of the manifold (4) and are the same as the ideal sliding mode equations (7)-(8). Like the systems with discontinuous control the equations of closed-loop motion do not depend on the control input and in the final stage of the motion both, ideal motion of the system with discontinuous control and motion of the system with continuous control are the same.

Control (12) for system (1)-(3) ensures the stability of the motion on the manifold $S$, i.e. the combination $\sigma_a(x)$ of the system’s states is forced to have value $\varphi(t)$. Any control problem, for system (1)-(3), which can be mathematically represented in the form of the sliding mode manifold (4), on which sliding mode can exist with control (5), can be solved applying control (12).

Generally speaking the reaching time in the system (14), (15) is infinite. The rate of decay is determined by matrix $D$ and the reaching is guarantied by the virtue of equation (15). In the system with discontinuous control the reaching time is finite but real sliding mode exhibits the motion in the $\varepsilon$-vicinity of the sliding mode manifold. If the motion of the system inside the $\varepsilon$-vicinity is accepted as the sliding mode motion, then system (14), (15) reaches this vicinity in the finite time determined by the matrix $D$. That is the basic difference from the systems with sliding modes. In the following sections the behavior of the systems with control (12) will be analyzed in more details and solutions without discontinuous term will be introduced. The discrete time realization of the control (12) is of the basic interest.
2.3 The discrete time realization of the control

The discrete time version of algorithms (11) and (12) can be written as

\[ u(kT) = \text{sat} \left[ u_{eq}(kT) + (B_2)^{-1}D\sigma(kT) \right] \]  

(16)

\[ u(kT) = \text{sat} \left[ u(kT - T) + (B_2)^{-1}D\sigma(kT) + \frac{d\sigma(kT)}{dT} \right] \]  

(17)

where \( \bullet(kT) \) denotes values at the moment \( t = kT(k = 1, 2, \ldots) \) and \( \bullet(kT - T) \) denotes the values at \( t = kT - T \), \( T \) is the sampling interval. With this control for a system without computational delay, equation (15) becomes

\[ \sigma(kT) = (E - TD)\sigma(kT - T) \]  

(18)

and if \( D \) is selected such that all eigenvalues of \( (E - TD) \) are within the unity circle the stability conditions are fulfilled. This system, like the system (14), (15) will, generally speaking, reach sliding mode manifold \( S \) in infinite time. If matrix \( D \) is selected diagonal with all elements equal \( d_{ii} = 1/T \) then (18) becomes

\[ \sigma(kT) = 0 \]  

(19)

and, after finite number of sampling intervals [4], sliding mode motion will occur. Further simplifications can be introduced by replacing \( d\sigma(kT)/dT \) by its first order approximation \( d\sigma(kT)/dT = (\sigma(kT) - \sigma(kT - T))/T \) to obtain

\[ u(kT) = \text{sat}[u(kT - T) + (B_2T)^{-1}(E + TD)\sigma(kT) - \sigma(kT - T)] \]  

(20)

This control does not require calculation \( d\sigma(kT)/dT \) and it is more suitable for implementation.

To illustrate the behavior of the system with proposed control strategies simulation results for a second order system

\[ \frac{d^2x_1}{dt^2} = 4(u - i_L(t)), \quad u \in [-20, 20], \quad x_2 = dx_1/dt \]  

(21)

\[ i_L(t) = \begin{cases} 5 & \text{for } 0 < t < 0.2s \\ 5 + 5\sin(25.84t) & \text{for } t > 0.2s \end{cases} \]

are depicted in Fig. 1 (a), (b) and (c). The sliding mode manifold is selected as the function of the control error and its time derivative

\[ S = \{x_1, x_2: C(x_1^{ref} - x_1) + (x_2^{ref} - x_2) = \sigma, \quad C = 100\}. \]

A step transient in \( x_1^{ref} = 0.01 \) [rad] at \( t = 0 \) is simulated. The control input is calculated according to the following expressions: \( u(kT) = 20\text{sig}(\sigma(kT)) \) in the case of discontinuous control action (diagrams (a) and (b) in Fig. 1 and \( u(kT) = \text{sat}[u(kT - T) + 0.25(d_{11}\sigma(kT) + d\sigma(kT)/dT)] \) in the case of the proposed algorithm, with \( d_{11} = 800 \).

Transients with discontinuous control are presented in Fig. 1 (a) and (b). In order to show influence of the sampling interval on the behavior of the closed loop system, sampling interval is selected to be \( T = 10^{-6} \) [s] in Fig. 1a, and \( T = 10^{-3} \) [s] in Fig. 1b. It can be observed that control error in Fig. 1b is higher and that the chattering clearly rises with the rise of the sampling interval. The activity of the control input is lower. Transients with continuous control algorithm (17) are depicted in Fig. 1c. In this case all simulations are done with a sampling interval \( T = 10^{-3} \) [s]. All other conditions are the same as for the discontinuous control case. As it can be observed all variables, including the control are smooth and chattering is indeed eliminated. The accuracy of the closed loop system is the same as for the systems with discontinuous control and 1000 times shorter sampling interval. The closed loop system exhibits the motion on the selected manifold as can be confirmed from the transients depicted in the phase plane. As it is expected, transients of the system with discontinuous control substantially
depend on the selection of the sampling interval. For properly selected sampling interval (Fig. 1a) transients are as it is theoretically predicted. Longer step size cause the low frequency chattering (Fig. 1b). It can be observed that the sliding mode function (and consequently $x_2$) has a ripple (the chattering) when discontinuous control is applied. With proposed control all transients are smooth and good load rejection and accuracy are achieved. In this examples no computational delay has been assumed and the calculation of the time derivative function $\sigma(x,t)$ has been required.

Fig. 1. Transients in the second order system with discontinuous (a), (b) and continuous control (c).

In Fig. 2 the behavior of the same system is studied against the change of the slope of the sliding mode manifold ($C$) and the decay of the Lyapunov function ($d_{11}$). These two quantities are the design parameters. Transients are depicted in the phase plane with control error on the horizontal axis and its derivative on the vertical axis and the time change in the Lyapunov function ($v$), the distance from sliding mode manifold ($\sigma$) and control error ($\Delta x_1$). In Fig. 2a the change of coefficient $d_{11}$ is selected to be $d_{11} = 2C, 4C$ and $8C$ respectively and $C = 15$. In Fig. 2b the value $d_{11} = 8C$ is kept while $C$ is changed to be $C = 5, 10$ and $15$. It can be observed that the reaching stage changes with the change of $d_{11}$ as it was predicted. The reaching stage is determined by the selection of the slope of the Lyapunov function and by its change the transients for both systems become very close. That can be seen better from the transients depicted in Fig. 2c and Fig. 2d where the change of the Lyapunov function and the control error are depicted for the same conditions as those in Fig. 2a and Fig. 2b, respectively.

The influence of the selection of the Lyapunov function decay is as it was theoretically predicted.

Fig. 2. The influence of the selection of the decay of the Lyapunov function on the transients of the system

3 Control of robotic manipulators

Trajectory tracking, force control and impedance control are common control problem in robotics. In the control system design these three problems are, usually, treated separately and then hybrid schemes are used to achieve the compliant control system. In the following sections we will show that algorithm developed in the previous section can be applied to all three problems and that the compliant control is natural extension in this framework. To examine the application of the developed algorithms to robotic manipulators control, first a suitable mathematical description will be derived and then, the switching manifolds will be formulated for all common problems.

3.1 The description of the control plant

A robotic manipulator actuated by AC electrical machines, is assumed as a control plant. It can be, in the Joint space, described by the following dynamical model

$$J \frac{d^2 \Theta}{dt^2} = g(\Theta, t) + Te$$  \hspace{1cm} (22)
where \( \Theta^T = [\Theta_1, ..., \Theta_n] \) is the vector of joint angular position; \( T_e^T = [T_{e1}, ..., T_{en}] \) is the electromagnetic torque vector usually treated as the control input; \( J \) is the inertia matrix. Disturbance \( g(\Theta, t) = [g_1, ..., g_n]^T \) includes all interconnected terms; \( J \) is diagonal positive definite matrix. The electromagnetic of each machine is defined, in the orthogonal frame of references related to the rotor flux, with the following set of the differential equations

\[
\frac{d^2 x}{dt^2} = J_{aco} \frac{d^2 \Theta}{dt^2} + \frac{dJ_{aco}}{dt} w = f_1(x, \dot{x}, t) + J_{aco} J^{-1} K_T i_q
\]

(25)

The Work space projection of the equations of motion can be derived from the direct kinematics \( x = F(\Theta) \), with a Jacobian determined as \( J_{aco} = [\partial F(\Theta)/\partial \Theta] \), [1], in the form

\[
\frac{d^2 z}{dt^2} = f(z, t) + B_i(i_d) i_q - B_i(i_d) i_L; \quad z \in \mathbb{R}^n, \quad f \in \mathbb{F}^n, \quad rank(B_i(i)) = n
\]

(26)

\[
\frac{di_j}{dt} = -L_j^{-1} R_j i_j + L_j^{-1} e_j(\Theta, i_j) + L_j^{-1} u_j, \quad (j = 1, ..., n)
\]

(27)

where \( u_j = [u_{dj} \ u_{jq}]^T \) is two dimensional control, \( i_j = [i_{dj} \ i_{jq}]^T \) is the current vector. The torque vector is defined as

\[
T_e = K_T(i_q) i_q; \quad i_q \in \mathbb{R}^n, \quad i_d \in \mathbb{R}^n
\]

(24)

\( K_T(i_d), L_j \) and \( R_j \) are diagonal positive definite matrices. The disturbance vector can be expressed as \( g(\Theta, t) = -K_T(i_d) i_L \). The subsystem (23) is mostly neglected in the description of the robotic manipulators [1]. For \( n = 1 \), (22), (23) and (24) describe the dynamics of an AC electrical machine.

The Work space projection of the equations of motion can be derived from the direct kinematics \( x = F(\Theta) \), with a Jacobian determined as \( J_{aco} = [\partial F(\Theta)/\partial \Theta] \), [1], in the form

\[
\frac{d^2 x}{dt^2} = J_{aco} \frac{d^2 \Theta}{dt^2} + \frac{dJ_{aco}}{dt} w = f_1(x, \dot{x}, t) + J_{aco} J^{-1} K_T i_q
\]

(25)

The description of the electromagnetic subsystem is not influenced by this transformation. Both, the Joint space and the Work space descriptions, can be rewritten in the following form

\[
\frac{dz}{dt} = f(z, t) + B_i(i_d) i_q - B_i(i_d) i_L; \quad z \in \mathbb{R}^n, \quad f \in \mathbb{F}^n, \quad rank(B_i(i)) = n
\]

(26)

\[
\frac{di_j}{dt} = -L_j^{-1} R_j i_j + L_j^{-1} e_j(\Theta, i_j) + L_j^{-1} u_j, \quad (j = 1, ..., n)
\]

(27)

where the meaning of the state vector \( z^T = [\Theta_1, ..., \Theta_n] \) or \( z^T = [x_1, ..., x_n] \) depends on the selection of the state space; matrix \( B_i = J^{-1} K_T \) in the Joint space and \( B_i = J_{aco} J^{-1} K_T \) in the Work space. Control input \( u_j(j = 1, ..., n) \) is discontinuous while the supply currents \( i_d \) and \( i_q \) are continuous.

Mathematical model (26) and (27) will be further referred to as "full order dynamics" of robotic manipulators. It describes the mechanical motion (26) with current \( i_q \) as the control input and electromagnetic subsystem with discontinuous control \( u \) as control input. It has been proven [8] that due to the sliding mode existence on the manifold \( S_i = \{\Theta, w, i: \sigma = i^{re} - i = 0\} \) the dynamics of the electromagnetic subsystem (27) can be reduced to the zero order system \( i^{re} = i \) (or \( i_d = i_d^{re} \) and \( i_q = i_q^{re} \)) thus reducing (26)-(27) to the model (26). Model (26) will be further refereed to as the "reduced order dynamics". The control system design will be demonstrated for both "full order" and "reduced order" dynamics of a robotic manipulator. In both cases the sliding mode approach will be applied. The "full order dynamics" has the discontinuous control inputs thus the classical design of sliding modes with discontinuous control [12] will be applied. The "reduced order dynamics" has continuous inputs thus the algorithms developed in the previous section will be applied.

### 3.2 Formulation of the control problems

The motion of the manipulator is solely determined by the position, velocity and acceleration vectors. Generally speaking, the control tasks should be formulated as a constrain which these three
coordinates must satisfy. This formulation determines the motion of mechanical subsystem, but does not include any requirement or restriction on the electromagnetic subsystem. For adopted model of the electromagnetic torque (24) with $K_T(i_d)i_q$ as the connecting term between mechanical and electromagnetic subsystems. The component $i_q$ should be determined from the mechanical motion constraints and the component $i_d$ should be determined from the requirement that $K_T(i_d)$ is kept constant or slow varying in all operational conditions of the actuators. This operation condition for the actuators can be achieved by the proper selection of the reference value of the current $i_d$ and selection of the discontinuous control $u_d$ to achieve the current tracking $i_d = i_d^{ref}$. With this in mind, further only the formulation of the mechanical motion requirements will be treated for both the "full order dynamics" and the "reduced order dynamics".

"Full order dynamics": The control goal is to find control which will force the continuous, generally speaking nonlinear vector function $\sigma_a(z, dz/dt, d^2z/dt^2)$ to track its reference value $\varphi(t)$, or in other words that the system’s state is forced to remain on the smooth manifold

$$S_z = \left\{ z, \frac{dz}{dt}, \frac{d^2z}{dt^2} : \varphi(t) - \sigma_a(z, \frac{dz}{dt}, \frac{d^2z}{dt^2}) = \sigma(z, t) = 0 \right\}$$

(28)

In this formulation the different control problems are defined by the selection of the function $\sigma_a(z, dz/dt, d^2z/dt^2)$. In Table 1 one possible selection for the trajectory tracking, the impedance control and the force control is presented. It is important to notice that, in order to be a sliding mode function, $\sigma(z, t)$ must be explicit function of the acceleration $d^2z/dt^2$.

The superscripts $^{ref}$ means a reference value, $^{ext}$ means external force. The trajectory tracking and the impedance control differs in the selection of the reference. Indeed for $C_1 = M^{-1}D$ and $C_2 = M^{-1}K$, both problems are defined by the same structure of the sliding mode manifold. Given sliding mode manifolds provide direct selection of the discontinuous control (the supply voltages of the electrical actuators).

TABLE 1. The selection of the sliding mode manifold for different control tasks (full order dynamics).

"Reduced order dynamics": If electromagnetic processes (27) are neglected or if the current control loop is designed separately then model (26) can be adopted as the description of a robotic manipulator. The currents are treated as the control inputs. The control goal is to find control which will force the continuous, generally speaking nonlinear vector function of the system’s states $\sigma_a(z, dz/dt)$ to track its reference value $\varphi(t)$, or in other words that the system’s state is forced to remain on the smooth manifold

$$S_z = \left\{ z, \frac{dz}{dt} : \varphi(t) - \sigma_a(z, \frac{dz}{dt}) = \sigma(z, t) = 0 \right\}$$

(29)

In Table 2 one selection of the sliding mode manifold for the trajectory tracking, the impedance control and the force control is presented. The meaning of variables and superscripts is the same as in Table 1. The manifold for impedance control remains the same as for the "full order dynamics". Other two tasks are defined by the manifold of the lower order.

The trajectory tracking, the force control and the impedance control are formulated in the same form, i.e. as a requirement that the system’s state are restricted to the selected manifold in the state space. This allows to conclude that the compliance control, as a task to track desired trajectory in compliance with sensed force, can be obtained by the selection of the appropriate manifold in the state space, on which the system’s motion must be confined. A transition from one manifold to another can be achieved on the points of the intersection of the manifolds.
4 Control system design

In this section the control input will be calculated for the tasks presented in the Table 1 and Table 2. The compliant control will be analyzed in the framework developed in the previous sections.

4.1 Full order dynamics

From (28), Table 1. and "full order dynamics" (26) and (27) it follows that discontinuous control should be selected as [7,14]

\[
    u_{dj}(z, i_d) = -\psi_{dj}(z, i_d, t) \sigma_{dj}(z, i_d, t) \text{ sig}(\sigma_{dj}(z, i_d, t)) \quad (30)
\]

\[
    u_{qj}(z, i_q) = -\psi_{qj}(z, i_q, t) \sigma_{qj}(z, i_q, t); \quad (j = 1, ..., n) \quad (31)
\]

Functions \(\psi_{dj}(z, i_d, t)\) and \(\psi_{qj}(z, i_q, t)\) are continuous and bounded. With control (30) and (31) the sliding mode exists on the intersection of the manifolds

\[
    S_d = \{ i_d : i_d^{ref} - i_d = \sigma_d(i_d, t) = 0 \} \quad (32)
\]

\[
    S_q = \left\{ z, \frac{dz}{dt}, \frac{d^2z}{dt^2} : \sigma_a \left( z, \frac{dz}{dt}, \frac{d^2z}{dt^2} \right) = \sigma_q(z, t) = 0 \right\} \quad (33)
\]

\[
    \sigma_q^T = [\sigma_{q1}, ..., \sigma_{qn}] \in \mathbb{F}^n, \quad \varphi^T = [\varphi_{q1}, ..., \varphi_{qn}] \in \mathbb{F}^n
\]

The sliding mode equations of motion become

\[
    i_d = i_d^{ref}(t) \quad (34)
\]

\[
    \sigma_q(z, t) = 0 \quad (35)
\]

\[
    \frac{d^2z}{dt^2} = f(z, t) + B_1(i_d)i_q - B_1(i_d)i_L \quad (36)
\]

For trajectory tracking and impedance control equations (34)- (36) reduces to (34) and (35). Indeed, from (35) acceleration can be calculated so, when sliding mode exists (35) and (36) reduces to the equation of the sliding mode manifold (35). That is very important because the selection of the sliding mode manifold is a design "parameter". For trajectory tracking the solution of the sliding mode equation (35) gives

\[
    \frac{d^2z}{dt^2} = C_1(z^{ref} - z) + C_2 \frac{d(z^{ref} - z)}{dt} + \frac{d^2z^{ref}}{dt^2} \quad (37)
\]

After some algebraic calculations the supply current can be expressed as

\[
    i_q = \text{ sat} \left[ i_L + B_i^{-1}(i_d) \left( C_1(z^{ref} - z) + C_2 \frac{d(z^{ref} - z)}{dt} + \frac{d^2z^{ref}}{dt^2} \right) \right] \quad (38)
\]

If sliding mode motion exists on the manifold (28), the current has the same value as one obtained from the "disturbance controller" [2]. This is very interesting result from which the equivalency of the
sliding mode design and the disturbance controller design is established. These two methods lead to the same solution for appropriate selection of the design parameters. If this current can be supplied to the actuators the motion of the system will remain on the sliding mode manifold (28) after reaching it, independently of the way the current source is designed. Thus a continuous input into system (26) can provide the motion along selected sliding mode manifold.

Such a design allows the elimination of chattering, because both $i^r_{qref}$ and acceleration (and consequently $i^a_{qref}$) are continuous functions. The discontinuous control is a structural property of the electrical actuators supply system, and it cannot be regarded as a consequence of the control system design, but contrary, the application of the sliding modes is a natural in this case.

4.2 The reduced order dynamics

The continuous control design is based on the dynamical description of a robotic manipulator (26), the control problem formulation as listed in Table 2 and the general results presented by the algorithms (11) and (17). The role of the mechanical motion controller design is to provide the reference input $i^r_{qref}$ for the current controller. The d-component reference current input is selected so matrix $K_T(i_d)$ is constant, or slow varying, over the range of operation and $i^r_{qref}$ must be selected to obtain desired closed-loop dynamics. By inspection of (1)-(3) and (26) follows that the application of algorithm (11) is straightforward and leads to the selection of the q-component of the reference current as

$$i^r_{qref} = \text{sat} \left[ i_L + (GB_i)^{-1} \left( D\sigma_q + \frac{d\sigma_q}{dt} - Gf \right) \right]; \quad G = \left[ \frac{\partial \sigma_q}{\partial z} \right]. \quad (39)$$

The algorithm (17) can be applied to obtain

$$i^r_{qref} = \text{sat} \left[ i_q + (GB_i)^{-1} \left( D\sigma_q + \frac{d\sigma_q}{dt} \right) \right] \quad (40)$$

The discrete time versions of the control algorithms become

$$i^r_{qref}(kT) = \text{sat} \left[ i_q(kT - T) + (GB_i)^{-1} \left( D\sigma_q(kT) + \frac{d\sigma_q}{dt}(kT) \right) \right] \quad (41)$$

$$i^r_{qref}(kT) = \text{sat} \left[ i_q(kT - T) + (GB_i)^{-1} \left( (E + TD)\sigma_q(kT) - \sigma_q(kT - T) \right) \right] \quad (42)$$

To know the acceleration is needed to implement algorithms (38), (39) and (40). Acceleration is not required for algorithm (42) and this algorithm can be regarded as more appropriate for application. This form of algorithm will be further used in presenting the simulation and experimental results for trajectory tracking, virtual impedance control, force control and compliant control schemes.

4.3 Trajectory tracking

The selection of the manifold and the control algorithms are presented in Table 3 for both the ”full order dynamics” and the ”reduced order dynamics”. In both cases only the mechanical motion controller is presented and it is assumed that the control of electromagnetic state of the actuators is separately designed. For ”full order dynamics” two form of algorithm are presented: one that includes the load current (the disturbance) information and another which includes the measurement of the actual
current. Sometimes even the current measurement can be avoided and the previous value of the controller output can be used instead. For "reduced order dynamics" the algorithm (41) and its approximation (42) are listed. The algorithm (42) is much simpler then these proposed in [4,5,6,10]. No acceleration nor the disturbance are required for its implementation. For reference current calculation the "gain matrix" $B_i$ can be selected to have so called nominal value and all parameter changes can be considered as part of the disturbance like it is treated in "disturbance controllers" [2]. To calculate the distance from the sliding mode manifold the position error and its derivative are required. To find control error derivative very simple estimation techniques can be used [7,14].

TABLE 3. The control algorithms for trajectory tracking

To show the validity of the proposed control scheme transients obtained on the experimental DD robotic manipulator (Fig. 3a) are presented in Fig. 3b and Fig. 3c. In Fig. 3b the phase plane transient in the step change of the second link reference position is depicted. The dynamics of the second joint position control loop is selected to follow the manifold $\sigma_{q2} = 15\Delta\theta + d\Delta\theta/dt = 0$. Algorithm (22) is applied with $d_{22} = 100$. The transient clearly shows that actual motion approaches the manifold as it is theoretically predicted and demonstrated in simulation (Fig. 2). In Fig. 3c several experiments are depicted. First time response for a experiment with step change in the second link reference is presented. Selected step in reference is selected 0.01 [rad] in order to demonstrate the accuracy of the system. The steady state error is zero. The step transient in the second joint position while third joint tracks a sinusoidal reference is depicted to show the robustness of the system against parameters variation. The cosine reference tracking in the second link is depicted in the last diagram. Smooth approach, small tracking error and smooth control current can be observed. That shows that chattering is indeed eliminated.

Fig. 3. The experimental robotic manipulator (a), the phase plane transient in the step change of the second joint position reference (b) and different transients of the second and third link.

4.4 Impedance control

The virtual mechanical impedance is defined as a linear combination of the manipulator’s state coordinates and corresponding sliding mode manifold is defined as

$$S_{imp} = \left\{ z, \frac{dz}{dt}, \frac{d^2z}{dt^2} : F^{ext} - M \frac{d^2x}{dt^2} + K \frac{dx}{dt} + Dx = \sigma(x, t) = 0 \right\}$$

(43)

The sliding mode manifold is, essentially the same as the one used in the trajectory tracking problem with a "full order dynamics". By solving $\sigma(x, t) = 0$ the reference current can be expressed as

$$i_{qref} = \text{sat} \left[ i_q + \left( MJ_{aco} J^{-1} K_T \right)^{-1} \left( Dx + K \frac{dx}{dt} + M \frac{d^2x}{dt^2} - F^{ext} \right) \right]$$

(44)

It is easy to confirm that this algorithm is the same as for the trajectory tracking.

4.5 Force control

To show the applicability of the proposed approach for the force control, the force developed in a contact with environment is modeled as was proposed in [15]. The contact force is described by the
following nonlinear model

\[ F_i = \begin{cases} 
   H_i(1 + k_i d \Delta z_{ei}/dt)(\Delta z_{ei})^{1.5}; & \text{if } \Delta z_{ei} > 0 \land (1 + k_i d \Delta z_{ei}/dt) > 0 \\
   0; & \text{otherwise}
\end{cases} \]

where \( z_e \) is the position of the obstacle. This model can be rewritten in as \( F^s = H(f_1(\Delta z) + f_2(\Delta z_e)d\Delta z_e/dt) \) for \( z > z_e \), \( \Delta z_e = z - z_e \), and \( F^s = 0 \) for \( z < z_e \). If the desired force is \( F^{ref} \) the sliding mode manifold should be selected as

\[ S_F = \{ x : \sigma_F = F^{ref} - F^s(\Delta z_e) \} \] (46)

As follows from (45) and (46) time derivative of the sliding mode function \( \sigma_F \) depends on control and consequently algorithm (41) can be applied. After some algebra control current can be expressed as

\[ i^{ref}_q(kT) = sat[i_q(kT) + (H_1(\Delta z)J_{aco}J^{-1}K_T)^{-1}T^{-1}[(E + TD)\sigma(kT) - \sigma(kT - T))] \] (47)

Diagonal matrix \( H_1(\Delta z) = Hf_2(\Delta z_e) \) is a time varying. Like in the trajectory tracking algorithm the elements of this matrix, for the control computation purposes, can be substituted by their "nominal values".

### 4.6 Compliance control

The trajectory tracking, the force control and the virtual mechanical impedance control differ from each other only in the definition of the control error expressed as the distance from the sliding mode manifold. The control algorithm, for all three control tasks, can be written in the form

\[ i^{ref} = sat[i + \Omega f(\sigma)] \] (48)

The matrix \( \Omega \) and function \( f(\sigma) \) must be selected according to the control problem i.e. in the proposed framework the compliance control means the change of the control error in the general algorithm (48).

To verify proposed algorithm the simulations on two D.O.F manipulator in vertical plane are presented in Fig. 4. The contact forces in x and y direction are modeled by (45) with \( H_x = 5000 \) and \( H_y = 10000 \) respectively, while controller is designed with \( H = 10000 \) in both directions. The sampling interval is selected \( T = 1ms \). The parameters of the manipulator are selected as \( m_1 = 0.5 \text{ kg}, m_2 = 1 \text{ kg} \) for \( t < 3s \) and \( m_2 = 4 \text{ kg} \) for \( t > 3s, l_1 = 1 \text{ m}, l_2 = 1 \text{ m} \). The position of x-direction obstacle is moving from the manipulator tip with velocity \( 0.1 \text{ m/s} \) e.g. \( x_a = 0.4 + 0.1t \) and y-direction obstacle is \( y_a = 1.35 \) for \( t < 2s \) and \( y_a = 0.9 \) for \( t > 2s \). The reference forces are selected to be constant in both direction. The coefficients in sliding mode manifolds are \( C_i = 20, d_{ii} = 100 \). This selection ensures that all possible combinations of compliant control the trajectory tracking, the combination of trajectory tracking in one direction and the force control in other and the force control in both directions. To verify the robustness against parameters change, a variation of the second link mass is selected during the interval while force control in both direction is active. The transient between trajectory tracking and force control is distinguished using very simple algorithm. \( \Omega f(\sigma) = max([\Omega f(\sigma)]_{position}, [\Omega f(\sigma)]_{force}) \) is selected in (48).

Fig. 4. The transients in a position and force control system for two D.O.F degrees of freedom robotic manipulator.

Observations of the transients depicted in Fig. 4 confirms all theoretical predictions. No overshoot in position tracking and in force control are observed. The influence of the parameter’s changes is observed to be small. Smooth transient between trajectory tracking and the force control is observed.
5 Conclusions

In this paper a general solution of the sliding mode control for systems linear with respect to control is proposed. Then this solution is applied to the trajectory following, the impedance and the force control problems in robotics. The proposed algorithm, based on the sliding mode motion on the selected manifolds in the state space, is derived from the selected Lyapunov functions and stability conditions. Derived control action is continuous thus any chattering is eliminated. Unlike other algorithm intended to avoid sliding mode chattering this approach uses only the information about distance from the sliding mode manifold to derive the control. It has been shown that this control, for intentionally selected manifolds, is equivalent to the acceleration control method. The application for robotic manipulators control is straight forward in both, the Joint space and the Work space, problems. Proposed solution leads to the single controller for all three control tasks: tracking, impedance control and force control. The only task dependent parameter is the equation of the sliding mode manifold.

References


10. S. V. Drakunov, and V. I. Utkin, ”On discrete-time sliding modes”, *Proc. of Nonlinear control*

12. V. I. Utkin, Sliding modes in control and optimization, (Springer-Verlag, 1992)


### TABLE 1. The selection of the sliding mode manifold for different control tasks (full order dynamics).

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>Joint Space</th>
<th>Work Space</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Position tracking</strong></td>
<td>$\sigma_a = C_1 \Theta + C_2 \frac{d\Theta}{dt} + \frac{d^2 \Theta}{dt^2}$</td>
<td>$\sigma_a = C_1 x + C_2 \frac{dx}{dt} + \frac{d^2 x}{dt^2}$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = C_1 \Theta_{ref} + C_2 \frac{d\Theta_{ref}}{dt} + \frac{d^2 \Theta_{ref}}{dt^2}$</td>
<td>$\varphi = C_1 x_{ref} + C_2 \frac{dx_{ref}}{dt} + \frac{d^2 x_{ref}}{dt^2}$</td>
</tr>
<tr>
<td></td>
<td>$C_1, C_2 &gt; 0$</td>
<td>$C_1, C_2 &gt; 0$</td>
</tr>
<tr>
<td><strong>Impedance control</strong></td>
<td>$\sigma_a = D \Theta + K \frac{d\Theta}{dt} + M \frac{d^2 \Theta}{dt^2}$</td>
<td>$\sigma_a = D x + K \frac{dx}{dt} + M \frac{d^2 x}{dt^2}$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = F^{ext}$</td>
<td>$\varphi = F^{ext}$</td>
</tr>
<tr>
<td></td>
<td>$M, K, D &gt; 0$</td>
<td>$M, K, D &gt; 0$</td>
</tr>
<tr>
<td><strong>Force control</strong></td>
<td>$\sigma_a = C_3 F^{ext} + \frac{dF^{ext}}{dt}$</td>
<td>$\sigma_a = C_3 F^{ext} + \frac{dF^{ext}}{dt}$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = C_3 F_{ref} + \frac{dF_{ref}}{dt}$</td>
<td>$\varphi = C_3 F_{ref} + \frac{dF_{ref}}{dt}$</td>
</tr>
<tr>
<td></td>
<td>$C_3 &gt; 0$</td>
<td>$C_3 &gt; 0$</td>
</tr>
</tbody>
</table>
TABLE 2. The selection of the sliding mode manifold for different control tasks (reduced order dynamics).

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>Joint Space</th>
<th>Work Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position tracking</td>
<td>$\sigma_a = C_1 \Theta + \frac{d\Theta}{dt}$</td>
<td>$\sigma_a = C_1 x + \frac{dx}{dt}$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = C_1 \Theta^{ref} + \frac{d\Theta^{ref}}{dt}$</td>
<td>$\varphi = C_1 x^{ref} + \frac{dx^{ref}}{dt}$</td>
</tr>
<tr>
<td></td>
<td>$C_1 &gt; 0$</td>
<td>$C_1 &gt; 0$</td>
</tr>
<tr>
<td>Impedance control</td>
<td>$\sigma = D \Theta + K \frac{d\Theta}{dt} + M \frac{d^2\Theta}{dt^2}$</td>
<td>$\sigma = D x + K \frac{dx}{dt} + M \frac{d^2x}{dt^2}$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = F^{\text{ext}}$</td>
<td>$\varphi = F^{\text{ext}}$</td>
</tr>
<tr>
<td></td>
<td>$M, K, D &gt; 0$</td>
<td>$M, K, D &gt; 0$</td>
</tr>
<tr>
<td>Force control</td>
<td>$\sigma_a = F^{\text{ext}}$</td>
<td>$\sigma_a = F^{\text{ext}}$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = F^{\text{ref}}$</td>
<td>$\varphi = F^{\text{ref}}$</td>
</tr>
</tbody>
</table>
1. Sliding mode manifold (Joint Space-full order dynamics):

\[
\sigma = C_1 (\Theta^\text{ref} - \Theta) + C_2 \frac{d(\Theta^\text{ref} - \Theta)}{dt} + \frac{d^2(\Theta^\text{ref} - \Theta)}{dt^2}; \quad C_1, C_2 > 0
\]

The reference current:

\[
i_q^{\text{ref}} = \text{sat} \left[ i_L + K_T^{-1} J \left( C_1 (\Theta^\text{ref} - \Theta) + C_2 \frac{d(\Theta^\text{ref} - \Theta)}{dt} + \frac{d^2(\Theta^\text{ref} - \Theta)}{dt^2} \right) \right];
\]

\[
i_q^{\text{ref}}(kT) = \text{sat} \left[ i_q(kT) + K_T^{-1} J J^{-1} \left( (E + TD)\sigma(kT) - \sigma(kT - T) \right) \right].
\]

2. Sliding mode manifold (Work Space-full order dynamics):

\[
\sigma_x = C_1 (x^\text{ref} - x) + C_2 \frac{d(x^\text{ref} - x)}{dt} + \frac{d^2(x^\text{ref} - x)}{dt^2}; \quad C_1, C_2 > 0
\]

The reference current:

\[
i_q^{\text{ref}} = \text{sat} \left[ i_L + K_T^{-1} J J^{-1} \left( C_1 (x^\text{ref} - x) + C_2 \frac{d(x^\text{ref} - x)}{dt} + \frac{d^2(x^\text{ref} - x)}{dt^2} \right) \right];
\]

\[
i_q^{\text{ref}}(kT) = \text{sat} \left[ i_q(kT) + K_T^{-1} J J^{-1} \sigma(x) \right].
\]

3. Sliding mode manifold (Joint Space-reduce order dynamics):

\[
\sigma = C_1 (\Theta^\text{ref} - \Theta); \quad C_1 > 0
\]

The reference current:

\[
i_q^{\text{ref}} = \text{sat} \left[ i_q + K_T^{-1} J \left( D\sigma + \frac{d\sigma}{dt} \right) \right];
\]

\[
i_q^{\text{ref}}(kT) = \text{sat} \left[ i_q(kT) + K_T^{-1} J a \sigma(x) \right].
\]

4. Sliding mode manifold (Work Space-reduce order dynamics):

\[
\sigma_x = C_1 (x^\text{ref} - x); \quad C_1 > 0
\]

The reference current:

\[
i_q^{\text{ref}} = \text{sat} \left[ i_q + K_T^{-1} J J^{-1} \left( D\sigma_x + \frac{d\sigma_x}{dt} \right) \right];
\]

\[
i_q^{\text{ref}}(kT) = \text{sat} \left[ i_q(kT) + K_T^{-1} J J^{-1} \sigma_x(kT) - \sigma_x(kT - T) \right].
\]
Figure 1: Transients in the second order system with discontinuous (a), (b) and continuous control (c).
Figure 2: The influence of the selection of the decay of the Lyapunov function on the transients of the system.
Figure 3: The experimental robotic manipulator (a), the phase plane transient in the step change of the second joint position reference (b) and different transients of the second and third link.
Figure 4: The transients in a position and force control system for two D.O.F degrees of freedom robotic manipulator.